Onsets of avalanches in the BTW model

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Abstract: The onsets of toppling and dissipation in the BTW model are studied by computer simulation. The distributions of these two onset times and their dependences on the system size are also studied. Simple power law dependences of these two times on the system size are observed and the exponents are estimated. The fluctuation of the average (spatial) height in the subcritical region is studied and observed to increase very rapidly near the SOC point.

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I. Introduction

There exists some extended driven dissipative systems in nature which show self-organised criticality (SOC). This phenomena of SOC is characterised by sponteneous evolution into a steady state which shows long-range spatial and temporal correlations. The concept of SOC was introduced by Bak et al in terms of a simple cellular automata model [1]. The steady state dynamics of the model shows a power law in the probability distributions for the occurence of the relaxation (avalanches) clusters of a certain size, area, lifetime, etc. Extensive work has been done so far to study the properties of the model in the steady SOC state[2-10]. Using the commutative property of the particle addition operator this model has been solved exactly [2]. Several properties of this critical state, e.g., entropy, height correlation, height probabilities, etc have been calculated analytically [2-3]. But the critical exponents have not been calculated analytically and hence an extensive numerical efforts have been performed to estimate various exponents [4-7]. The values of critical exponent for size and lifetime distributions of avalanches starting at the boundary have been calculated [8]. Recently, the avalanche exponents were estimated using the renormalization scheme [9].

Very few efforts are made to study the systematic evolutions of the system towards the SOC state. In the subcritical region the response of a pulsed addition of particles has been studied and it has been observed [11] that the ratio of response time (δt) to the perturbation time (Δt) diverges as the system approaches the critical state (i.e., $R = \frac{\delta t}{\Delta t} \sim (z_c - z)^{-\gamma}$), which implies the critical slowing down in this model. Grassberger and Manna [12] also anticipated such kind of scaling behaviour ($<s>, <t> \sim (z_c - z)^{-\beta}$) and obtained only for two dimension.

In this paper, we have focussed on the subcritical region of the time evolution of the BTW model. We have studied how the onset time of toppling and that of dissipation (escape of particle through the boundary) vary with the system size. The distributions of these two onset times have also been found. We have also studied the growth of fluctuations of the height variable (averaged over all sites) as the system attains the critical state.

II. The model and simulation

The BTW model is a lattice automata model which shows some important properties of the dynamics of the system which evolves spontaneously into a critical state. We consider a two dimensional square lattice of size $L \times L$. The description of the model is the following: At each site (i, j) of the lattice a variable (so called height) $z(i, j)$ is associated which can take positive integer values. In every time step, one particle is added to a randomly chosen site according to

$$z(i, j) = z(i, j) + 1. \qquad (1)$$

If, at any site the height variable exceeds a critical value z_m (i.e., if $z(i,j) \geq z_m$), then that site becomes unstable and it relaxes by a toppling. As an unstable site topples, the value of the height variable, of that site is decreased by 4 units and that, of each of the four of its neighbouring sites increased by unity (local conservation), i.e.,

$$z(i,j) = z(i,j) - 4 \tag{2}$$

$$z(i,j \pm 1) = z(i,j \pm 1) + 1 \quad \text{and} \quad z(i \pm 1,j) = z(i \pm 1,j) + 1 \tag{3}$$

for $z(i,j) \geq z_m$. Each boundary site is attached to an additional site which acts as a sink. We use here the open boundary conditions so that the system can dissipate through the boundary. In our simulation, we have taken $z_m = 4$. The main investigations in this paper can be divided as follows:

(1) Studies regarding the onset times of the toppling and dissipations, where we allow the system to evolve under the BTW dynamics (following eqns 1-3) starting from an initial condition with all the sites having $z = 0$. With the evolution of time, the height at different sites first increases due to random addition of particles. As soon as the height at any site reaches (or exceeds) the maximum value ($z_m = 4$), that site topples. We call this time, when the toppling starts in the system, the onset time of toppling, T_o^t. In most of the cases, the interior sites first topple and then, after some time toppling occurs at the boundary sites. The system starts to dissipate (through boundary) as soon as any boundary site topples. The time, taken by the system to start dissipation, is called onset time of dissipation, T_o^d. The onset of toppling of the boundary site (to start dissipation) can occur in either of the two ways: (i) via the primary avalanche of the boundary site, (ii) through the secondary avalanche followed by a primary avalanche, initiated at any interior site of the lattice. These two onset times of the system may change appreciably as one change the random sequence of addition of particles. As a result, these two times may have wide fluctuations in their distributions. We have studied here the statistical distributions of these two times, T_o^t and T_o^d. The size (of the system) dependences of these two onset times are also studied.

(2) Studies regarding the growth of fluctuations near the SOC point: The average (spatial) value of z, i.e.,

$$\bar{z} = (1/N) \sum_{i=1}^{N} z_i \qquad\qquad (N = L^2)$$

increases almost linearly with the time in the subcritical region and then in the critical state, $\bar{z}$ becomes steady apart from some fluctuations [4]. Grassberger and Manna [12] have studied the system size dependence of the fluctuation of $\bar{z}$ in the critical state. Here, we pay some attention to study how the fluctuation of $\bar{z}$ changes as the

system approaches SOC for a particular length ($L = 100$). Here, the fluctuation of $\bar{z}$ means the standard deviation of $\bar{z}$, at a particular time, for different random sample, i.e.,

$$\delta z = \sqrt{\frac{1}{N_s}\sum_{l=1}^{N_s}(\bar{z}_l - < \bar{z} >)^2}$$

where $\bar{z}_l$ is the value of $\bar{z}$ for l^{th} random sample and $< \bar{z} >$ is the average value of $\bar{z}$ calculated from N_s number of different random samples, i.e.,

$$< \bar{z} >= (1/N_s)\sum_{l=1}^{N_s} z_l.$$

III. Results

In our simulation, for a fixed system size ($L = 100$), the distribution of T_o^t and T_o^d are obtained from 10^4 different samples. Figure 1 shows the distribution of the onset time of toppling ($D(T_o^t)$) and the distribution of the onset time of dissipation ($D(T_o^d)$). It has been observed that the onset times for toppling and that for dissipation vary in a wide range but they have well defined symmetric distributions. The width of these two distributions differ appreciably, where it is much much larger in the case of dissipations. Since the number of boundary sites are smaller than that of the interior sites the width of the distribution of T_o^d is much larger than that of T_o^t.

The onset time (average) for toppling and that for dissipation will depend upon the linear size (L) of the system. The log-log plot of these two onset times are depicted in Fig. 2. The simulation results show that these dependences are power law type. The exponents are also estimated within limited accuracy. $T_o^t \propto L^a$ and $T_o^d \propto L^b$, where $a \sim 1.48$ and $b \sim 1.70$. All these data are obtained by averaging over 100 different samples for $30 \leq L \leq 300$.

The growth of fluctuation of $\bar{z}$ is plotted against the time of evolution of the system in the subcritical region in Fig. 3. It shows that the fluctuation increases very sharply near the critical point (SOC). The fluctuations of $\bar{z}$ are calculated for $L = 100$ using 400 random different samples.

IV. Summary

We studied here, when the toppling and the dissipation start in the BTW model. There is a well defined symmetric distribution of the onset time for toppling and that for dissipation. These two onset times depend (power law) on the system size with different exponents.

¿From Fig.1, we see that the maximum value of the onset time for dissipation is of the order of L^2, when the average value of the hight variable is of the order of unity (i.e., $\bar{z} \sim 1$). Thus, there is a very little chance, that onset of toppling of the

boundary site occurs due to the secondary avalanches of the interior site's avalanches. Almost all the topplings of the boundary sites, for the onset of dissipation, are due to the avalanches initiated at the boundary. In this sense, onset of dissipation can be considered as onset of toppling of the boundary site (due to the avalanche, initiated at that boundary site).

It is also important to know whether the system size dependences of these two onset times are same in the subcritical region for the other SOC models show the power law.

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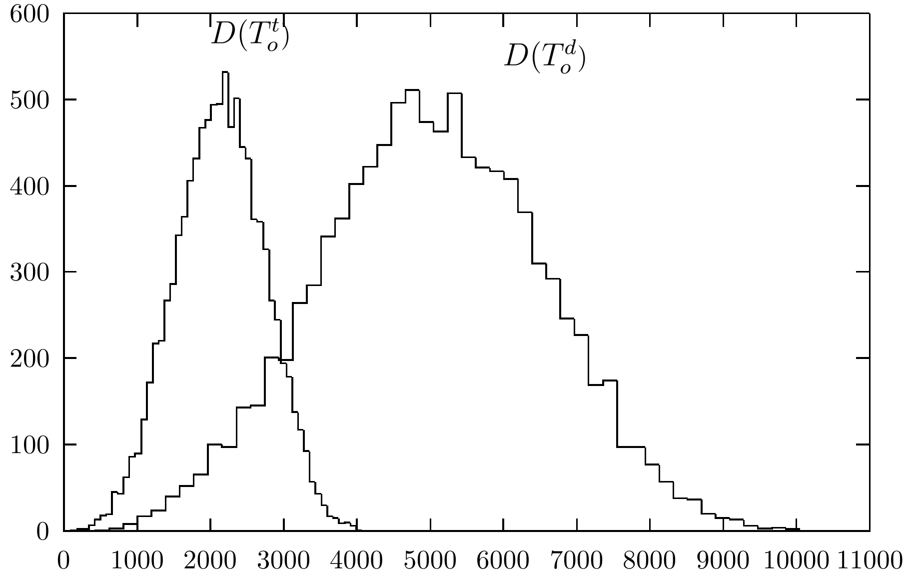

Fig. 1. The unnormalized distributions of onset time of toppling (T_o^t) and the onset time of dissipation (T_o^d).

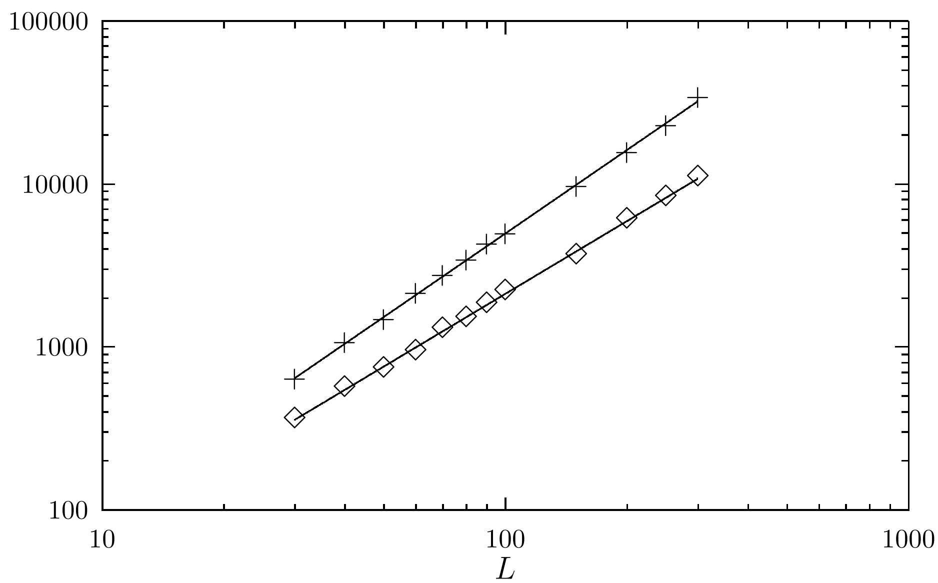

Fig. 2. The onset times, for toppling (T_o^t) ($\diamond$) and dissipation (T_o^d) (+), are plotted against the system size L in log-log scales. The solid lines are linear best fit.

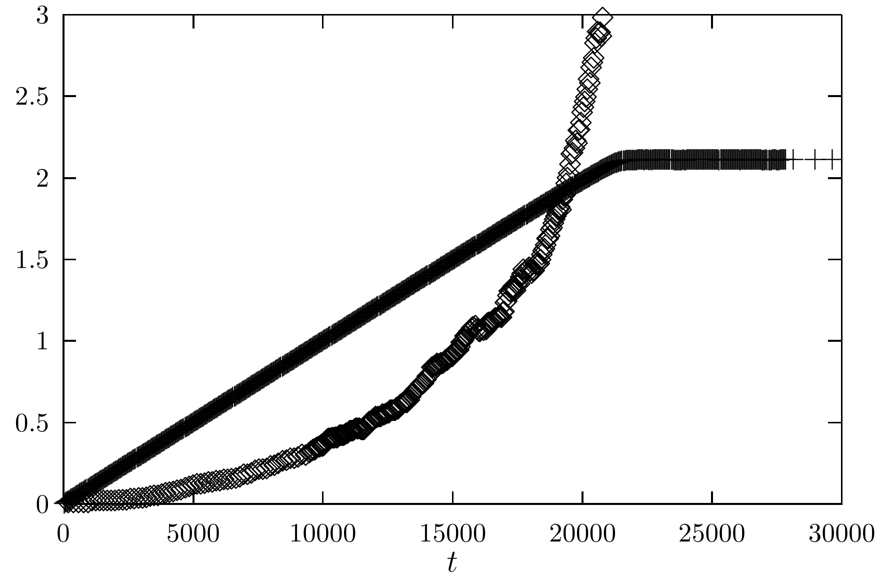

Fig. 3. The time varations of $< z >$ (+) and $\delta z \times 10^3$ ($\diamond$). At SOC $< z > = 2.124$ (Ref. [4]).